A Simple Model of Cascades in Networks*

Yongwhan Lim† MIT
Asuman Ozdaglar‡ MIT
Alexander Teytelboym§ University of Oxford

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Abstract

We consider a linear threshold model of cascades in networks. An agent switches (e.g. adopts an innovation) if the proportion of his neighbors who have already switched exceeds his threshold. Agents' thresholds are drawn randomly at the start of the cascade process. We present a result for the expected number of switches in arbitrary finite networks with any initial seeds. We define a new measure of an agent's ability to influence a cascade in a given network, called *cascade centrality*, which is the *expected size* of the cascade when the agent is the only seed in the network. We then define *contagion centrality*, which is the probability that *all agents* switch when the node is a seed. For certain network topologies, we find analytic expressions for cascade centrality and contagion centrality and show that there may be tension between them. Yet nodes with high cascade and contagion centrality share an interesting property: they have many neighbors, but their neighbors have few neighbors. As an illustration of cascade centrality, we look at how the network structure affects optimal prices when a profit-maximizing firm tries to spread an innovation. Optimal price and seeding behavior can be counterintuitive when firms price irreversible cascades. Our tractable model can be extended in various ways.

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†Operations Research Center and Laboratory for Information and Decision Systems, MIT. Email: yongwhan@mit.edu.

‡Department of Electrical Engineering and Computer Science and Laboratory for Information and Decision Systems, MIT. Email: asuman@mit.edu.

§Institute for New Economic Thinking at the Oxford Martin School, University of Oxford. Email: alexander.teytelboym@inet.ox.ac.uk.

1 Introduction

Many phenomena that occur in social and economic networks are, at least temporarily, irreversible. Examples include the spread of incurable diseases (Klovdahl, 1985), informational leaks and rumors (Moreno et al., 2004), systemic risk of bank failures (Elliott et al., 2014), platform adoption (David, 1985), drug addiction (Bauman and Ennett, 1996), patenting (Aghion et al., 2015), religious conversion (Stark and Bainbridge, 1980), and dropping out of high school (Staff and Kreager, 2008). These phenomena have two key features. First, they exhibit network externalities: agents are (heterogeneously) affected by their neighbors in the network. The structure of the network is critical for the irreversible phenomena that exhibit network effects. Some processes remain contained in isolated clusters and others spread to the whole network. Secondly, these phenomena are inherently path-dependent: their irreversibility means that early history matters for the overall outcome. Whether the goal is to reduce contagion risk or to maximize adoption of an innovation, understanding how early history and network structure affect cascades is important for good policies.

In order to analyze cascades in networks, we revisit the familiar linear threshold model introduced by Granovetter (1978). We call an irreversible transition to new state, such as adoption of a product, a *switch* (Jackson, 2008, p. 295). Initially, all agents in the network are switched off. Then some agents are randomly (or optimally) switched, i.e., seeded. Every agent in the network is endowed with an individual threshold. In the following periods, once the *proportion* of neighbors that switches exceeds his threshold, the agent also switches (Granovetter, 1978; Schelling, 1978).[1] This process propagates through the network, but once an agent switches, he remains switched forever.[2] Our objective in this paper is to characterize how the structure of the network and the initial seed set affect the total number of switches.

In contrast to some of the previous work (Acemoglu et al., 2011; Yildiz et al., 2011), we do not look at a particular instance of a distribution of thresholds. Instead we assume that agents' thresholds are randomly and independently drawn from the uniform distribution at the start of the cascade (Kempe et al., 2003). This is a reasonable assumption if the social planner has no reason to believe that some thresholds are more

[1] There is empirical evidence that adoption products, such as iPhone, is sensitive to the proportion of friends who have adopted it (Godinho de Matos et al., 2014). The linear threshold model may not be an ideal model for a variety of processes, such as learning about new products, where the *number* rather than the *proportion* of neighbors affects the probability of switching. For example, Banerjee et al. (2013) do not find an effect of the proportion of neighbor-adopters on the adoption of microfinance.

[2] Kleinberg (2007) and Adam et al. (2012) call these types of threshold models "progressive".

likely than others (by the principle of insufficient reason). We show that under this assumption the probability that an agent switches equals to the average probability that his neighbors switch conditional on his not switching. This striking simplification allows us to derive a simple analytical expression for the probability of any agent switching, building on insights by Kempe et al. (2003). We introduce a new concept – *cascade centrality* of a node – which measures the expected size of a cascade when this node alone is a seed in a network. Our key theorem states that the cascade centrality of an agent is equal to his degree (plus one) less the contribution of loops (paths that bend back on themselves exactly once). Although calculating the number of paths or loops is computationally challenging, it offers us a lot of analytical tractability. For certain networks, such as trees, cycles, complete networks, and a large class of random graphs, we can give analytical expressions for cascade centrality. In a tree, the cascade centrality of any node is its degree (number of neighbors) plus one, whereas in a complete network the cascade centrality of any node is proportional to the square root of the number of nodes. We then introduce another concept called *contagion centrality* of a node which measures the probability that the cascade reaches every other node in network when this node is the only seed (Morris, 2000). We show that there is a tension between cascade centrality and contagion centrality. For example, while the cascade centrality of a node in a complete network increases at the rate of $\sqrt{n}$ (where n is the number of nodes), the probability of contagion declines slowly at the rate $\frac{1}{n}$.

The theoretical literature on cascades and diffusion in networks is vast. Irreversibility of our cascade process sets out the present paper apart from the strand of the literarture which assumes that agents can switch multiple times (Blume, 1993; Ellison, 1993; Blume, 1995; Young, 2006; Montanari and Saberi, 2010; Adam et al., 2012). These papers usually assume that agents play a coordination game with their neighbors and analyze the dynamics using tools from evolutionary game theory.[3] For certain problems, such as the possibility of contagion, the models are essentially equivalent (Morris, 2000; Watts, 2002; Dodds and Watts, 2004; Lelarge, 2012; Adam et al., 2012). Our approach allows us to analyze precisely how the structure of the network affects the cascade process. Since the number of paths determines expected switches, we show that, at least theoretically, we cannot expect any straightforward comparative statics

[3]In evolutionary models, multiple switches can occur either because agents make an error or because their thresholds are redrawn during the cascade. The process is ergodic so the stationary distribution does not depend on the initial conditions. Our model can be seen as a coordination game with heterogeneous payoffs, but in which seeds have a dominant strategy to switch. Another interesting class of non-ergodic models is one in which the seeds can also switch back and forth, but where the thresholds are fixed and the agents always best-respond. Adam et al. (2012) show that in this class of models, the process either converges to an equilibrium or the system oscillates between exactly two states.

using most macroscopic network properties, such as clustering (Centola et al., 2007; Centola, 2010; Acemoglu et al., 2011).[4]

We illustrate cascade centrality with an economic application. We consider how a profit-maximizing firm tries to diffuse a product in a network. It needs to pick a seed and a price. Higher prices make agents less likely to adopt the product for any proportion of neighbors who have already adopted it. While this problem becomes analytically challenging, we are able to use our tools to derive some interesting propositions. We show that the structure of the network dramatically affects the optimal price. For example, in a line network the optimal price is bounded away from zero as the line becomes long, whereas when a complete graph becomes large the optimal price goes to zero. While there have been several papers on optimal static pricing in social networks (Candogan et al., 2012; Campbell, 2013; Ajorlou et al., 2015), to the best of our knowledge, our work is the first one to analyze optimal pricing and seeding of cascades simultaneously in general finite networks. Among papers that focus on analytical results, Candogan et al. (2012) consider optimal pricing in a model with a divisible good and Campbell (2013) considers pricing in random networks. In Lim et al. (2014), we use cascade centrality to analyze competitive rumor spread.

Many strategic aspects of cascades in networks, such as attacks on networks (Acemoglu et al., 2013) and the role of imperfect information in collective action (Bikhchandani et al., 1992; Banerjee, 1992), are beyond the scope of this paper.

We proceed as follows. Section 2 describes the "simple" linear threshold model and the dynamics of the cascade. Section 3 explains the role of the uniform threshold assumption, describes the analytical expression for the expected number of switches, introduces cascade centrality and states the main theorem linking cascade centrality to the number of loops. This section also introduces contagion centrality. Section 4 applies the main theorem and describes analytical results for cascade centrality and contagion centrality for certain topologies. In Section 5, we describe the application to optimal pricing. Section 6 briefly mentions a few extensions of the model, while Section 7 concludes and points to directions for further work. All the proofs and simulations are in the Appendix.

[4]In Appendix C, we show that higher clustering can both increase and decrease the number of expected switches. This is also true when we consider "average cascades" i.e. when the seed is random.

2 General model of cascades

2.1 Preliminaries

Let $G(V, E)$ be a simple (unweighted and undirected), connected graph with a set of n agents $V := \{1, \ldots, n\}$ and a set of m links E.[5] We denote the neighbors of $i \in V$ as $N_i(G) := \{j | (j, i) \in E\}$ and the degree of i as $d_i := |N_i(G)|$. A *threshold* for agent i is a random variable Θ_i drawn independently from a probability distribution with support $[0, 1]$. The associated multivariate probability density function for all the nodes in the graph is $f(\boldsymbol{\theta})$. Each agent is $i \in V$ assigned a threshold θ_i. Let's define the threshold profile of agents as $\boldsymbol{\theta} := (\theta_i)_{i \in V}$. A network $G_{\boldsymbol{\theta}}$ is a graph endowed with a threshold profile.

2.2 Dynamics of a deterministic cascade

First, let us consider dynamics of a deterministic cascade on a given network $G_{\boldsymbol{\theta}}$. We follow the exposition of Acemoglu et al. (2011).[6] The binary state of agent i at time t is denoted $x_i(t) = \{0, 1\}$, corresponding to "off" and "switched". Denote by $S_t(G_{\boldsymbol{\theta}})$ the set of additional switches in network $G_{\boldsymbol{\theta}}$ at time t. At time $t = 0$, a subset of agents $S_0 \subseteq V$ is selected to be the seed set. We assume that at $t = 0$ agents switch if and only if they are in the seed set. Hence, at $t = 1$, any $i \in V \setminus S_0(G_{\boldsymbol{\theta}})$ will switch, i.e., $i \in S_1(G_{\boldsymbol{\theta}})$ if

$$\frac{|S_0(G_{\boldsymbol{\theta}}) \cap N_i(G_{\boldsymbol{\theta}})|}{|N_i(G_{\boldsymbol{\theta}})|} \geq \theta_i.$$

This means that at $t = 1$ agents switch only if the proportion of their neighbors who were seeds exceeds their threshold. Then, for a given period $t \geq 0$ and node $i \in V \setminus \cup_{\tau=0}^{t-1} S_\tau$ will switch at t, i.e., $i \in S_t(G_{\boldsymbol{\theta}})$ if

$$\frac{|\{\cup_{\tau=0}^{t-1} S_\tau(G_{\boldsymbol{\theta}})\} \cap N_i(G_{\boldsymbol{\theta}})|}{|N_i(G_{\boldsymbol{\theta}})|} \geq \theta_i.$$

This means that any agent who has not switched by some period t, switches in time period $t + 1$ if the proportion of his neighbors who switched is greater or equal to

[5] An extension of the model to a directed graph is fairly straightforward and does not substantially affect the analysis.

[6] The dynamics of our model can be described as modified local interaction game (Morris, 2000, Equation 2.1) in which critical probability q (payoff from being switched off) is drawn from Θ for each agent, except the seeds for which $q = \theta = 0$ i.e. being switched is a best response. In a Nash equilibrium, each agent best responds in his own-payoff local interaction game.

his threshold θ_i. For a given network G_θ, define the fixed point of the process as $S_0 = S(G_{\boldsymbol{\theta}}, S_0) \Rightarrow S_t(G_{\boldsymbol{\theta}}) = \emptyset$ for all $t > 0$. As Acemoglu et al. (2011) show, this fixed point always exists.

2.3 Expected size of a cascade

Let us now consider the "average" cascade dynamics on a network when thresholds are drawn from $f(\boldsymbol{\theta})$. For a given graph G and S_0, we can map each realization $f(\boldsymbol{\theta})$ to a set of switches $S(G_\theta, S_0)$. Hence, we can treat $S(G_\theta, S_0)$ as a random variable with a probability distribution $f(\boldsymbol{\theta})$. Let us compute the expected probability of any particular agent i switching in network G is given seed S_0 by taking the expectation with respect to $f(\boldsymbol{\theta})$

$$\mathbb{P}_i(G, S_0) = \int_{\mathbb{R}^n} |S(G_{\boldsymbol{\theta}}, S_0) \cap \{i\}| f(\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

Hence, the expected number of switches in graph G is:

$$\mathbb{E}[S(G, S_0)] := \int_{\mathbb{R}^n} |S(G_\theta, S_0)| f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \sum_{i=1}^{n} \mathbb{P}_i(G, S_0).$$

While our deterministic process mimics the one in Acemoglu et al. (2011), our focus on the expected size of a cascade by integrating over possible thresholds is similar to Kempe et al. (2003). In Section 6, we also show that our analysis can be extended to the case where agents do not switch immediately and have multiple switching attempts.

3 Analysis of a simple model

3.1 Uniform distribution

We first consider how the probability of an agent switching depends on the probability of his neighbors switching. We show that if the threshold for this agent is drawn from uniform distribution then his probability of switching is simply equal to the average switching probability of his neighbors conditional on his *not* switching.

Proposition 1. *Let $\{G(n)\}_{n\in\mathbb{N}^+}$ be a set of networks in which $d_i(G(n)) = n - 1$ and a non-empty seed set is $S_0 \subseteq V \setminus \{i\}$ on each $G(n)$, then*

$$\mathbb{P}_i(G(n), S_0) = \sum_{j \in N_i(G(n))} \frac{\mathbb{P}_j(G(n), S_0 | i \notin S)}{d_i(G(n))}$$

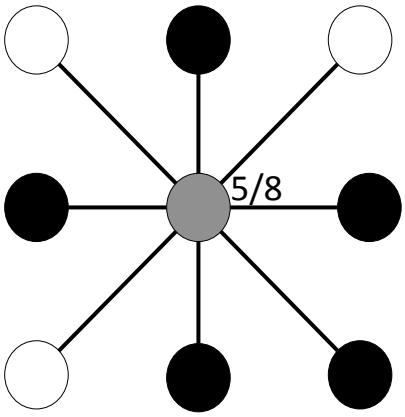


Figure 1: Uniform Threshold Rule for a Star

for any seed set on every $G(n)$ *if and only if* Θ_i *is uniformly distributed on* $[0,1]$ *almost everywhere.*

It is easy to illustrate the proposition with a example of star network, which also places a key role in the proof. Figure 1 shows that five of center's neighbors have a probability of switching equals to 1 because they are seed. The other three have a probability of switching equal to zero, conditional on the center not switching. Hence, the probability of the center node switching is $\frac{5}{8}$ whenever his threshold is drawn from the uniform distribution. In the rest of the paper we use the following assumption.

Assumption 1. For any $G_{\boldsymbol{\theta}}$ and every $i \in V$, $\Theta_i \sim \mathcal{U}(0,1)$ and independent.

We therefore drop $\boldsymbol{\theta}$ subscript and henceforth $G \equiv G_{\boldsymbol{\theta}}$. From now on, we use graph/network, agent/node, and link/edge interchangeably.

It may be tempting to conclude that the uniform distribution of thresholds implies that any agent's probability of switching is equal to the probability of switching of all agent's neighbors. However, this is not the case precisely due to the path dependence of the switching process. Consider a line network of length three with one seed at a leaf node. It is easy to show that if thresholds are drawn from a uniform distribution, the probability of the center and other leaf node switching is $\frac{1}{2}$. However, the switching probability of the middle node is *not* the average of the probabilities of his neighbors (which is $\frac{3}{4}$).

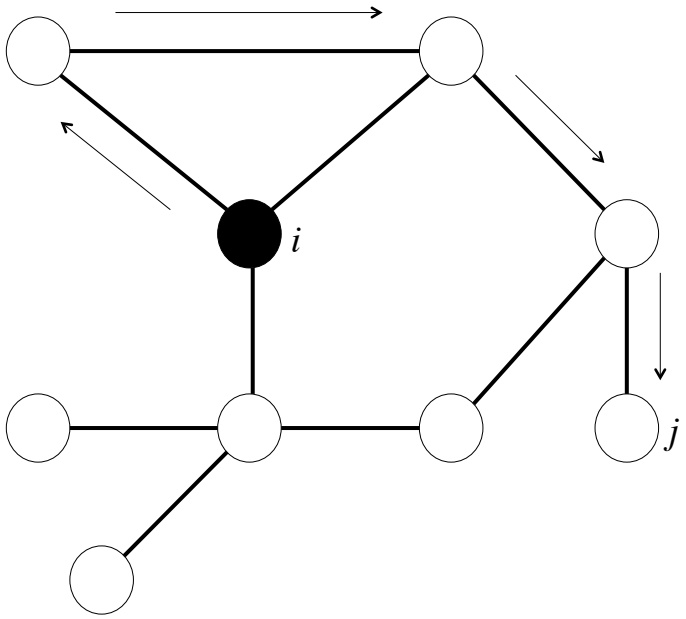


Figure 2: A path between two nodes

3.2 Cascade centrality

In order to capture path dependence of the cascade process, we first introduce paths.[7]

Definition 1. A sequence of nodes $P = (i_0, \cdots, i_k)$ on a graph G is a *path* if $i_j \in N_{i_{j-1}}(G)$ for all $1 \leq j \leq k$ and each $i_j \in P$ is distinct.

Since every node in the path is distinct, so is every edge. In general, there may be multiple paths between any two nodes. Figure 2 shows one possible path between nodes i and j. There are three paths between i and j in total in this network.

Definition 2. For a path P in G, the *degree sequence* along P is denoted $(d_i(G))_{i \in P}$. The *degree sequence product* along P is

$$\chi_P := \prod_{i \in P} \frac{d_i(G)}{d_{i_0}}.$$

The degree sequence product of a particular path multiplies the degree every node in the path except the first one. The degree sequence product of the path in Figure 2 is $2 \times 3 \times 3 \times 1 = 12$.

[7]Some graph theory textbooks refer to "paths" as "simple paths". Since we only use "simple paths" in this paper, we refer to them as "paths" without any ambiguity.

For any G and S_0, let $\mathcal{P}_{ji}$ be the set of all paths beginning at $j \in S_0$ and ending at $i \in V \setminus S_0$ and $\mathcal{P}^*_{ji} \subseteq \mathcal{P}_{ji}$ denote the subset of those paths that exclude any other node in S_0. The following result follows from the equivalence of the linear threshold process and random selection of "live-edge paths" established by (Kempe et al., 2003, Claim 2.6).

Proposition 2. *Given a graph G and seed S_0, the probability that node $i \in V \setminus S_0$ switches is*

$$\mathbb{P}_i(G, S_0) = \sum_{j \in S_0} \sum_{P \in \mathcal{P}^*_{ji}} \frac{1}{\chi_P}.$$

Proposition 2 provides a remarkable insight into the calculation of the size of the expected cascade in any network with any seed. It says that the expected probability of any node switching is equal to the sum of the degree sequence products along all the paths from each seed (avoiding any other seed) to the node. In Section 4, we show this result can be applied to analyze a variety of networks.

Figure 3 illustrates how to apply the proposition to a fairly general network with two seeds A and B. Each path is labelled in a different color (some paths may include paths in themselves). The numbers next to the nodes denote the probabilistic contributions – inverse of the degree sequence products – of various paths. Note that seed B does not affect the probability of switching of the nodes in the top triangle and of the leftmost node because any path from B to these nodes must pass through A.[8]

Using Proposition 2, we can define the expected size of the cascade when a particular node is the seed.

Definition 3. *Cascade centrality* of node i in graph G is the expected number of switches in that graph given i is the seed, namely

$$\mathcal{C}_i(G) := \mathbb{E}[S(G, \{i\})] = 1 + \sum_{j \in V \setminus \{i\}} \mathbb{P}_j(G, \{i\}) = 1 + \sum_{j \in V \setminus \{i\}} \sum_{P \in \mathcal{P}_{ij}} \frac{1}{\chi_P}$$

and the average cascade in a graph G is

$$\mathcal{C}(G) := \frac{\sum_{i \in V} \mathcal{C}_i(G)}{n}.$$

Cascade centrality captures the importance of a node in the network by measuring how large a cascade it induces when it alone is the seed. In order to see how cascade centrality can be used to generate insights into cascade processes, we prove a useful

[8] In the language of Bayesian networks, we could say that A is a *Markov blanket* for these nodes.

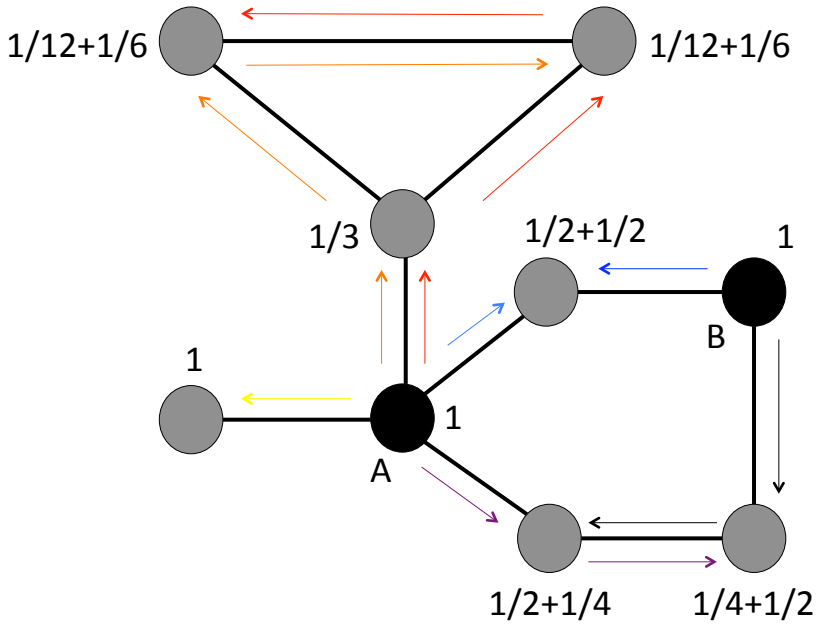


Figure 3: Network with two seeds and all the relevant paths

decomposition result. We first introduce a loop, which is a sequence of nodes (i.e. a walk) that bends back on itself exactly once.

Definition 4. A sequence of nodes $L = (i_0, \ldots, i_k)$ on a graph G is a *loop* if $(i_0, \ldots, i_{k-1})$ is a path and $i_k \in \{i_0, \ldots, i_{k-2}\}$ for some $k \geq 2$.

The degree sequence product χ_L for a loop $L = (i_0, \ldots, i_k)$ is equal to the degree sequence product of its corresponding path $(i_0, \ldots, i_{k-1})$ (see Definition 2). Let $\mathcal{L}_{ij}$ be the set of all loops beginning at node i and ending at node j. Figure 4 illustrates a loop beginning at node i.

Theorem 1. *The cascade centrality of any node i in G is:*

$$\mathcal{C}_i(G) = 1 + d_i - \sum_{j \in V} \sum_{L \in \mathcal{L}_{ij}} \frac{1}{\chi_L}$$

Theorem 1 shows that cascade centrality of any node in any network is at most its degree plus one. Fixing a node's degree, the presence of loops strictly reduces its cascade centrality. Long loops and loops that involve nodes with high degrees will reduce cascade centrality by less.

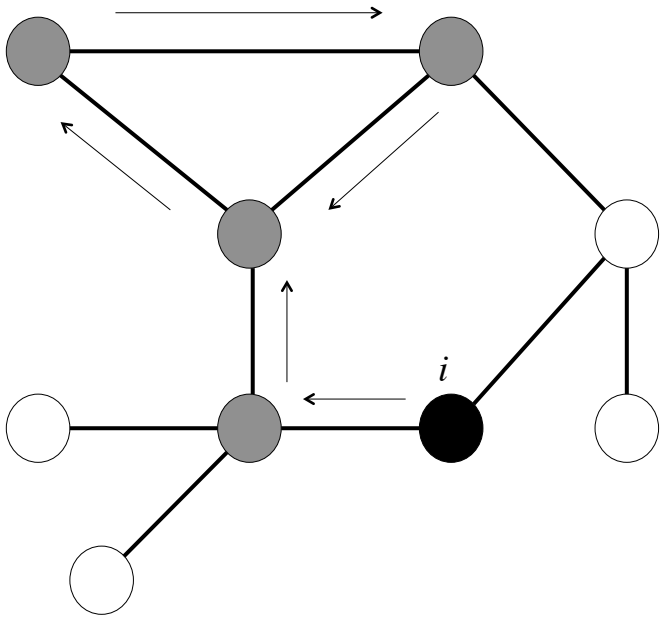


Figure 4: Loop

The theorem immediately suggests how to design a network that maximizes the maximum cascade centrality over all nodes in the graph. Let $\mathcal{G}$ be the set of all graphs with vertex set V ($|V| = n$) and $\mathcal{G}^m$ be the set of all graphs with vertex set V and exactly m edges ($|E| = m$). A *star* graph of order n is an acyclic graph (tree) on n nodes with one node having degree $n-1$ and the other $n-1$ nodes having degree 1.

Proposition 3. *Let*

$$G^* \cong \arg\max_{G\in\mathcal{G}}\{\max_{i\in V}\mathcal{C}_i(G)\}$$

Then, G^* is uniquely a star graph of order n.

Let

$$G^*_m \cong \arg\max_{G\in\mathcal{G}^m}\{\max_{i\in V}\mathcal{C}_i(G)\}$$

Then

$$\max_{i\in V}\mathcal{C}_i(G^*_m) > \max_{i\in V}\mathcal{C}_i(G^*_{m-1}) \text{ whenever } 1 \le m \le n-1$$

and

$$\max_{i\in V}\mathcal{C}_i(G^*_m) > \max_{i\in V}\mathcal{C}_i(G^*_{m+1}) \text{ whenever } n \le m \le \binom{n}{2}$$

Proposition 3 says that adding edges to and removing edges from a star graph can only reduce the cascade centrality of the node with the highest cascade centrality. Moreover, it seems clear that the ideal node for seeding has *high* degree, but all of its neighbors (and the neighbors of their neighbors) have a *low* degree. In contrast to the literature on "key players", nodes with high cascade centrality do not have influential friends - instead they have many unpopular friends.

Theorem 1 and our simulations (in the Appendix) also show that cascade centrality is closely related to degree centrality. However, the simulations also indicate cascade centrality can be very different from centrality notions based on counting the number of walks (i.e. sequences of connected nodes which are not necessarily distinct), such as eigenvector centrality and Katz-Bonacich centrality (Katz, 1953; Bonacich, 1987). Indeed, our simulations make it clear that since the contribution of paths of greater length decreases geometrically in cascade centrality, even knowing the set of paths of lengths one, two, and three can provide an excellent approximation to the expected size of the cascade.

3.3 Contagion centrality

In the previous section, we focused on the expected size of the cascade given a seed. We now turn our attention to extreme outcomes: i.e. the probability that the cascade reaches **every** node in the network (or a given connected subnetwork). This would allow us to compare how susceptible to total contagion different networks can be.

Definition 5. *Contagion centrality* of a node i is

$$\mathcal{K}_i(G) = \int_{\mathbb{R}^n} \mathbf{1}_n \left(|S(G_{\boldsymbol{\theta}}, \{i\})| \right) f(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

(where $\mathbf{1}_n$ is the indicator function taking value $\mathbf{1}$ when the argument is equal to n and zero otherwise) i.e. the probability that the entire network switches when i is the only seed.

Contagiousness of a graph G is

$$\mathcal{K}(G) = \frac{\sum_{i \in V} \mathcal{K}_i(G)}{n}$$

i.e. the probability that every agent in G switches when one agent is seeded uniformly at random.

Our key result in this section states that contagion centrality of a node depends on the product of the eigenvalues of the Laplacian matrix (the difference between the diagonal degree matrix and the adjacency matrix) and the degree of all other nodes rather than, like cascade centrality, directly on the degree of the node itself. The reason for the relationship between the Laplacian and contagion is that contagion (in any finite network) can be represented by a spanning tree and the product of the eigenvalues of the Laplacian counts the number of spanning trees in the network.

Theorem 2. *Contagion centrality of a node i in G is*

$$\mathcal{K}_i(G) = \frac{\frac{1}{n}\lambda_1 \times \ldots \times \lambda_{n-1}}{\prod_{j \in V \setminus \{i\}} d_j}$$

and contagiousness of a network is

$$\mathcal{K}(G) = \frac{\left[\frac{1}{n}\lambda_1 \times \ldots \times \lambda_{n-1}\right] \times \left[\sum_{j \in V} d_j\right]}{n \times \prod_{j \in V} d_j}$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are non-zero eigenvalues of the Laplacian matrix of G.

Unlike cascade centrality, computing contagion centrality for any node in any network only involves finding the eigenvalues of the Laplacian (and is therefore computationally tractable). Since contagion centrality only depends on the count of spanning trees and the degree sequence of the graph, comparative statics results can be derived straightforwardly from corresponding comparative statics on the number of spanning trees in the graph.[9]

4 Analytical results

4.1 Cascade centrality

We now show how Proposition 2 and Theorem 1 can be used to derive analytical expressions for the expected number of switches in certain fixed and infinite networks.

Corollary 1. *1. Suppose that G is a tree. Then for any $i \in V$,*

$$\mathcal{C}_i(G) = d_i(G) + 1$$

[9]See, for example, Das et al. (2013) for an analytical upper bound on the number of spanning trees in the graph as a function of various network parameters, such as the number of vertices, the number of edges, maximum degree, second maximum degree, and so on.

$$\mathcal{C}(G) = 1 + \frac{2(n-1)}{n}$$

2. *Suppose that G is a cycle of order n. Then, for any $i \in V$,*

$$\mathcal{C}_i(G) = 3 - \frac{1}{2^{n-2}}$$

Consider $\{G(n)\}_{n\in\mathbb{N}^+}$, a sequence of cycles of order n. Then

$$\lim_{n\to\infty} \mathcal{C}_i(G(n)) = 3$$

3. *Suppose that G is a complete graph of order n. Then, for all $i \in V$,*

$$\mathcal{C}_i(G) = \mathcal{C}(G) = 1 + (n-1)\left(\sum_{i=1}^{n-1} \mathbf{P}(n-2, i-1)\left(\frac{1}{n-1}\right)^i\right)$$

where $\mathbf{P}(n,i) \equiv \frac{n!}{(n-i)!}$ is number of ways of obtaining an ordered subset of i elements from a set of n elements.

Consider $\{G(n)\}_{n\in\mathbb{N}^+}$, a sequence of complete graphs of order n. Then

$$\lim_{n\to\infty} \frac{\mathcal{C}_i(G(n))}{\sqrt{n}} = \sqrt{\frac{\pi}{2}}$$

Our first result, an immediate corollary of Theorem 1, says that the expected number of switches from a single seed node in *any* tree equals to the degree of the seed plus one. The tree is a particularly convenient topology to analyze because there is a unique path between any two nodes i.e. there are no loops. Hence, the probability of switching of any node simply equals to the degree sequence product along this path. From this we obtain that the size of cascade starting from a random node in a (connected) tree is *independent of its structure*. Figure 5 illustrates the probabilities of switching for all the nodes in a tree with one seed.

Due to their symmetry, the cycle and the complete graph are also easy to analyze. In these networks, the identity of the seed node is irrelevant. As Figure 6 shows, in a cycle with one seed, there are only two paths to each node around the cycle. Alternatively, using Theorem 1, we can simply subtract the contribution of the unique loop from three since $d_i + 1 = 3$ for any node in a cycle. As the cycle becomes large, its behavior approaches that of a non-edge seed node (i.e. with degree 2) on a path.

In a complete graph, any node can be reached from any other node is one step. But, as Figure 7 shows, we must also keep track of the number of paths of every possible

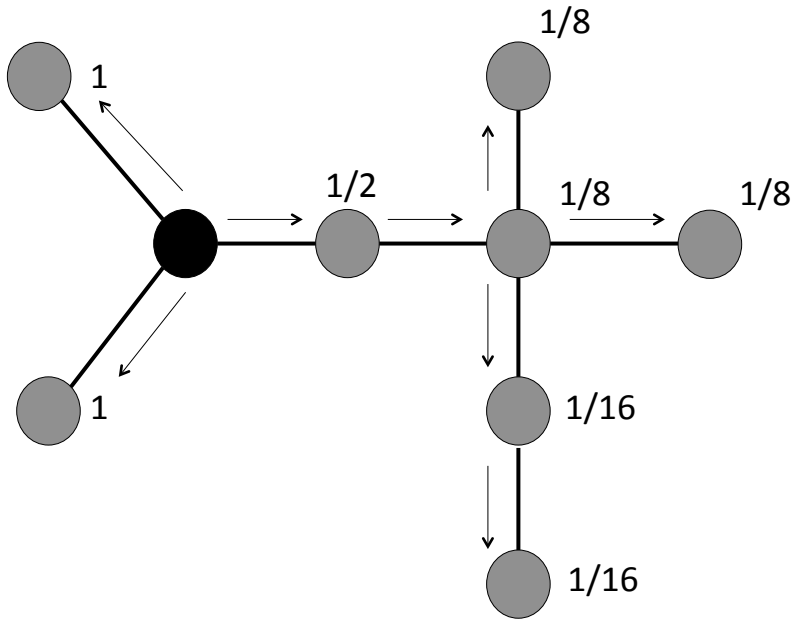


Figure 5: Tree with one seed

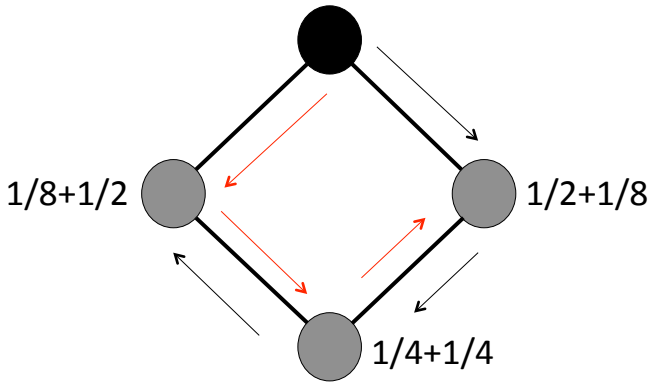


Figure 6: Cycle with one seed

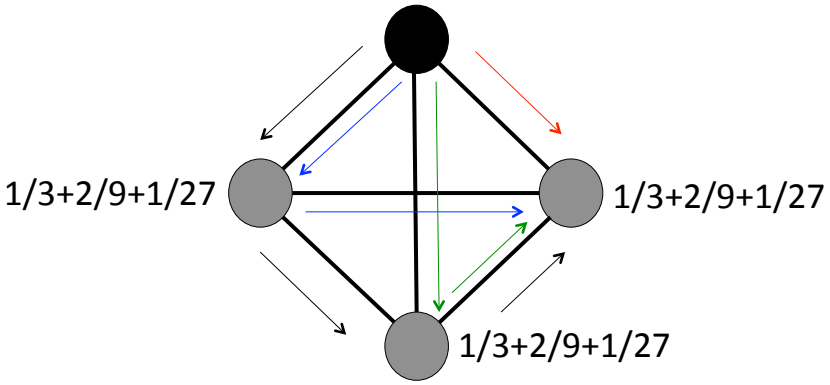


Figure 7: Complete graph with one seed

length. Fortunately, the number of paths of a length k can the expressed as number of ways to select k out of $n-1$ objects without repetition (for example, lottery numbers). The result also shows that the size of the expected cascade in a complete graph grows at the rate $\sqrt{n}$.

We now prove a general result for cascade centrality for large random graphs. In order to do this, we first introduce the configuration model, closely following Bollobás and Riordan (2015). The configuration model allows us to create random graphs with a given degree distribution. First, we generate a desired degree sequence. Then we pick randomly two elements of the degree sequence – nodes with "stubs" – and link them together. We then delete those two elements from the degree sequence. This may produce self-loops and multi-edges, but it can be shown that the degree distribution in the configuration model converges in probability to the desired degree distribution. To avoid excess notation, we delegate the technical description of what we call a "regular" configuration model $G^*_{\mathbf{d}_n}$ to the Appendix and simply state the key result.

Proposition 4. *Let $G^*_{\mathbf{d}_n}$ be a regular configuration model and consider the cascade centrality of a node in $G = G^*_{\mathbf{d}_n}$ chosen uniformly at random, then*

$$\Pr(\mathcal{C}(G) = 1 + \mathbb{E}(D)) \to 1$$

as $n \to \infty$ where $\mathbb{E}(D)$ is the expected degree of $G^*_{\mathbf{d}_n}$.

The configuration model is useful because it can be used to generate any random

graph model, including Erdős-Rényi graphs and power-law graphs. Our result only relies on the fact that the expected degree of the random graph is finite as the network gets large. The startling result (using Corollary 5 in Bollobás and Riordan (2015)) is that any random graph generated from the configuration model behaves like a tree.

Finally, we turn to the analysis of infinite lattices, first considered in a cascade framework by Morris (2000). We now turn our attention to infinite network. Note that under the uniform random threshold assumption, cascade centrality is well-defined for any infinite lattice with a fixed degree. While we cannot obtain analytical expression for cascade centrality in these lattices, we can give reasonable bounds. We will consider *hypercubic lattices* of dimension r: in these networks, nodes are the integer points of $\mathbb{R}^r$ and edges are the unit line segments between the integer points. Hypercubic lattices include the square lattice ($r = 2$) and hexagonal lattice ($r = 3$) discussed in Morris (2000).

Proposition 5. *Let G be an infinite hypercubic lattice of dimension $r \geq 2$, then for a generic node i:*

$$3 \leq \mathcal{C}_i(G) \leq 1 + 2r$$

The upper bound in this proposition comes from Theorem 1. The classical square lattice gives us us $3 \leq \mathcal{C} \leq 5$ (computationally, we get 3.5 up to paths of length 10), whereas for the hexagonal lattice we obtain tighter bounds of $3 \leq \mathcal{C} \leq 7$ (computationally, we get 5.0 up to paths of length 10).[10]

4.2 Contagion centrality

We now turn to the analysis of contagion centrality. As we pointed out, contagion centrality is a straightforward computational problem on finite networks. The following result illustrates the networks already covered by Corollary 1.

[10] A path on an infinite lattice is called a "self-avoiding walk" (Madras and Slade, 1993). "The enumeration of self-avoiding walks ... has for half a century, been among the most challenging problems in enumerative combinatorics." (Guttmann and Conway, 2001). The complexity of the problem of calculating the number of self-avoiding walks on an infinite lattice is not known (but conjectured to be NP-hard). Very few analytical results are known, Duminil-Copin and Smirnov (2012) is a notable exception. Using Theorem 2 and the results on the number self-avoiding walks up to length 10 from Fisher and Sykes (1959) we calculate that the cascade centrality of a node in a: honeycomb lattice is 2.913; square lattice is 3.498; Kagome (trihexagonal) lattice is 3.242; triangular lattice is 4.039. Here lattices with higher degree have higher cascade centrality, but lattices with higher clustering (and the same degree i.e. square and Kagome) have lower centrality.

Corollary 2. *1. Let G be a tree. For any $i \in V$*

$$\mathcal{K}_i(G) = \frac{1}{\prod_{j \in V \setminus i} d_j}$$

and

$$\mathcal{K}(G) = \frac{2(n-1)}{n \times \prod_{j \in V} d_j}$$

2. Let G be a cycle. For any $i \in V$

$$\mathcal{K}_i(G) = \mathcal{K}(G) = \frac{n}{2^{n-1}}$$

3. Let G be a complete graph. For any $i \in V$

$$\lim_{n \to \infty} \frac{\mathcal{K}_i(G)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\mathcal{K}(G)}{\frac{1}{n}} = e$$

Nodes with high cascade and contagion centrality have a lot of neighbors, who have fewer neighbors. However, our results highlight the tradeoff between cascade centrality and contagion centrality in networks. This should not be surprising. Cascade centrality depends *directly* on the degree of the node (Theorem 1) whereas contagion centrality does not. Let's compare trees and complete graphs. In trees, the maximum cascade centrality of a node is n. Cascade centrality of any node in a large complete graph, on the other hand, is roughly $\sqrt{n}$. However, contagion centrality declines exponentially fast in trees, it declines at rate $\frac{1}{n}$ is complete graphs. Therefore, while possible cascades are "smaller" in complete graphs, the probability of an extreme outcome i.e. a contagion is relatively greater. In this sense, our framework supports observations that complete graphs are, in some sense, "robust but fragile" (Acemoglu et al., 2015).

Finally, we can use contagion centrality to express the probability of switching for *any subset* of nodes that are reachable from the seed.

Proposition 6. *Consider a cascade process on $G(V, E)$. The probability that a connected set of nodes $V' \subset V$ switches when $i \in V'$ is the seed is*

$$\prod_{j \in V \setminus V'} \left[1 - \frac{|N_j(G) \cap V'|}{d_j}\right] \left[\frac{\frac{1}{n}\lambda_1 \times \ldots \times \lambda_{n-1}}{\prod_{j \in V' \setminus \{i\}} d_j}\right]$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are non-zero eigenvalues of the Laplacian matrix of G', subgraph induced by V' and the degrees of the nodes are calculated from G.

The first term expresses the probability that "none of the nodes that border with

V' switch". So, for example, no neighboring leaves can remain not switched. If none of the nodes that border with V' switch, this creates a "blanket" between V' and the $V \setminus \left[V' \cup_{j \in V'} N_j(G)\right]$ and ensures that none of these remaining nodes switch. The last term is simply the probability that all of V' switch that we derive using Theorem 2.

5 Application: Pricing cascades

In this section, we apply our simple model of cascades to pricing. Suppose that the seeder is in fact a profit-maximizing firm that is trying to spread its product. The firm selects one agent $i \in V$ as the seed and picks a uniform price $0 \leq \rho \leq 1$ for its product. The seed adopts the product immediately and the firm makes no profit off him. We assume that subsequent agents adopt the product at period $t+1$ if:

$$[1-\rho] \times [\text{fraction of neighbors who adopted by } t] \geq \theta_i.$$

When $\rho = 0$ we recover our simple linear threshold model and when $\rho = 1$ no agent, except the seed, adopts the product. More generally, each agent's demand for the product is downward sloping: The higher the price $\rho = 1$, the less likely (compared to the simple linear threshold model) is any agent to adopt the product for a given threshold.

When the seed is i and the price is ρ, the firm's profit is

$$\pi(i,\rho) = \rho \times \left[\sum_{j \in V \setminus \{i\}} \sum_{P \in \mathcal{P}_{ij}} \frac{(1-\rho)^{|P|}}{\chi_P} \right]$$

where the term in brackets is the "stochastic" cascade centrality, which we will return to later, and $|P|$ is the length of path P. The profit-maximization problem for the firm is now a lot harder: it needs to pick the optimal node i^* and the optimal price ρ^* simultaneously i.e. it is no longer sufficient to simply pick the node with the highest cascade centrality in order to maximize the size of the cascade. In general, the node with the highest cascade centrality is not the node that will maximize profit. Moreover, analytically solving the pricing problem for a general finite network is not possible. Nevertheless, our setting can provide a number of illuminating observations.

Proposition 7. *In any network and for any seed, the optimal price satisfies* $\rho^* \leq \frac{1}{2}$.

First, we easily show that the highest possible profit for a network of size n can be obtained in a star where is optimal price and seed are always the same.

Proposition 8. *Maximum profit in any network with n nodes is $\frac{1}{4}(n-1)$ and it is obtained in a star of order n. The optimal price is $\frac{1}{2}$ irrespective of the order of the star.*

Next, we consider a line network. Recall that every node in a line, except the leaves has a cascade centrality of 3. Our next result shows that the strictly optimal seed is, in fact, a penultimate node. Moreover, we can rank all the nodes according to their profitability.

Proposition 9. *Consider a line network G with ordered nodes $\{1, 2, 3, ..., n\}$*

1. *The optimal seeds on any line are nodes $i^* \in \{2, n-1\}$. As $n \to \infty$, $\rho^* \to \approx .465571$ and $\pi(i^*, \rho^*) \to \approx .418588$.*
2. *$\pi(i, \rho^*) > \pi(j, \rho^*) > \pi(1, \rho^*) = \pi(n, \rho^*)$ for any $2 \leq i < j \leq \lfloor n/2 \rfloor$*
3. *As $n \to \infty$ and $i \in \{1, n\}$, $\rho^* \to \sqrt{2} - 1$ and $\pi(1, \rho^*) \to (\sqrt{2} - 1)^2$.*
4. *For any seed, maximum profit is increasing in n.*

The reason for this result follows the same intuition as for nodes that have high cascade centrality. The firm profits the most from seeding the node that has a lot of neighbors, but also one whose neighbors have few neighbors. In the case of the line, this distinction is stark and analytically tractable. In cycles, the optimal seeding problem becomes symmetical and is similar to seeding a leaf on a line.

Proposition 10. *Consider $\{G(n)\}_{n \in \mathbb{N}^+}$, a sequence of complete cycles of order n. Then for a generic seed i*

$$\rho^* \to \sqrt{2} - 1$$

and

$$\pi(i, \rho^*) \to 1$$

as $n \to \infty$.

We can apply this to a cycle of 3 to get $\rho^* = \frac{4}{3} - \frac{\sqrt{7}}{3} \approx 0.451$ and $\pi(i, \rho^*) = \frac{1}{27}\left(7\sqrt{7} - 10\right) \approx 0.3156$. While the optimal price in a cycle and in a complete graph of thee is the same, optimal pricing in complete graphs looks radically different as the grow.

Proposition 11. *Consider $\{G(n)\}_{n \in \mathbb{N}^+}$, a sequence of complete graphs of order n. Then for a generic seed i*

$$\rho^* \to 0$$

and

$$\pi(i, \rho^*) \to 1$$

as $n \to \infty$.

The complete graph result can, in fact, be generalized to an r-regular tree.

Proposition 12. *Consider an r-regular infinite tree. Then for a generic seed i*

$$\rho^* = \frac{1}{1 + \sqrt{r}}$$

and

$$\pi(i, \rho^*) = \frac{r}{(\sqrt{r} + 1)^2}$$

It should be obvious that the last two results are completely congruent. As the regularity of the infinite tree increases, its optimal price tends to zero and its profit tends to 1, just like in a complete graph. The general, testable predictions of this model is that optimal price should fall in denser and larger networks, but the profit should rise. However, our results also highlight the subtle effects of the network structure on pricing cascading innovations.

6 Extensions

Although the model we presented in this paper is rather stylized, it provides a lot of insight into the effect of network structure on cascades. We now turn to several ways to enrich and parameterize both the simple and the competition model for empirical applications (see, for example, Hodas and Lerman (2014)).

6.1 Homophily

In our simple model all agents were identical. However, in many social networks there is substantial agent heterogeneity beyond the network structure. For example, neighbors can have heterogeneous impact on the probability of switching. This can be captured easily by directed, weighted networks Kempe et al. (2003). On the other hand, agents with similar thresholds might be more likely to be neighbors – this phenomenon is known as homophily McPherson et al. (2001). Our model can be extended to capture homophily in the following way: assign a parameter $\zeta_i \in (0, 1)$ to each agent in such a way that "similar" agents have similar ζ. The homophily parameter linearly scales

down the probability of switching given any number of neighbors who have switched. In order to adjust the cascade centrality measure, it is simply required to replace d_i with $\zeta_i d_i$ in every degree sequence product that involves agent i. All the results can go through with that amendment.

6.2 Susceptibility and network formation

Suppose that we are interested in the probabiliy that a node switches but we do not know where the cascade process begins. An application of this could be a which agent expect a cascade following a period of network formation (e.g. Blume et al. (2011); Erol and Vohra (2014); Farboodi (2014)). To keep things simple, suppose that one node is seeded randomly. Then we can define the susceptibility of i in G as

$$\phi_i(G) = \frac{\sum_{j \in V} \mathbb{P}_i(G, \{j\})}{n}$$

remembering that $\mathbb{P}_i(G, \{i\}) = 1$. It turns out that the susceptibility of any node is related to the cascade centrality of all other nodes. Let us define for convenience $\chi^{P_{ii}} = 1$.

Proposition 13. *Susceptibility of node i is*

$$\phi_i(G) = \frac{1}{d_i n} \left[\sum_{j \in N} \sum_{P_{ij} \in \mathcal{P}_{ij}} \frac{d_j}{\chi^{P_{ij}}} \right]$$

Moreover, in any regular graph G

$$\phi_i(G) = \frac{\mathcal{C}_i(G)}{n}$$

While the relationship between susceptibility and centrality can be quite complicated (even in trees where susceptibility of a node depends on the structure of the whole tree), in regular graphs, such as cycles or complete graphs, they are perfectly related: graphs (of the same size) in which nodes have higher cascade centrality are more susceptible.[11]

Let us briefly consider how susceptibility changes the outcomes of simple network formation games. Consider an extension of the setting of Jackson and Wolinsky (1996) (similar to Blume et al. (2011)) in which agents form a pairwise stable network in the first period and in the second period a cascade starting from a random seed hits the

[11] In general regular graphs, nodes will have different cascade centrality.

network. The agents receive a benefit from the network only if they do not switch ("fail") after the cascade, however, the network formation costs are sunk.

We can write the expected utility $u_i(G)$ of an agent in G

$$u_i(G) = (1 - \phi_i(G)) \sum_{j \neq i} \delta^{t_{ij}} - cd_i$$

where t_{ij} is the geodesic distance between i and j so the cost and benefits of links are exactly as in Jackson and Wolinsky (1996). Recall that when $\phi_i(G) = 0$ for all i there are conditions on c and δ under which star networks are stable (Jackson and Wolinsky, 1996). However, when agents anticipate cascades, we get a contrasting result:

Proposition 14. *A star is never stable in the network formation game with cascades.*

The simple proof shows that any two leaves in a star can add a link between them without increasing their probability of failure. To prevent them from doing that we require that $\delta < c$. However, the centre can also add and remove links without affecting his probability of failure, but in order to encourage him to keep his links we require that $\delta \geq c$.

This model can be developed further to find conditions under which complete networks are stable and under what conditions stable networks are inefficient (in general they will be). Since there are many options for specifying the utility function and the structure of the cascades, we leave this for future work.

7 Conclusions

This paper provided a simple framework for analyzing cascades in networks. We showed that when agents' thresholds are drawn from a uniform distribution the cascade process can be expressed as a count of paths attenuated by their degree sequence products.

We offered two new tools for the analysis of cascades in networks: cascade centrality (that measures the expected) and contagion centrality. We show that there is a tradeoff between the two measures. We then applied our models to a setting of cascades pricing.

Our simple model can also be used to explore other questions, such as network formation in the presence of cascade and optimal network design. The model can also be taken to data. We offered some guidance on possible parameterization in Section 6. Parameters δ, c and ζ can, in principle, be estimated. There are several obvious directions for future work. First, one could look at dynamic, rather than static, pricing (Ajorlou et al., 2015). Second, it would be informative to understand how network

formation would play out in rich economic settings under cascade and contagion threats described by our model (Babus, 2013; Erol and Vohra, 2014; Farboodi, 2014).

Appendix A: Proofs

Proofs of Propositions in Section 3

Proof of Proposition 1. First, we prove the following lemma, illustrated in the main text.

Lemma 1. *Let $\{G(n)\}_{n\in\mathbb{N}^+}$ be a sequence of star networks of order $n \in \mathbb{N}^+$ in which i is the center and the seed set is $S_0 \subseteq V \setminus \{i\}$, then*

$$\mathbb{P}_i(G(n), S_0) = \frac{|S_0|}{d_i(G(n))}$$

for any seed $0 \leq |S_0| \leq n-1$ on every $G(n)$ if and only if Θ_i is uniformly distributed on $[0,1]$ almost everywhere.

Proof of Lemma 1. The "if" direction is trivial since for any star $G(n)$ and $0 \leq |S_0| \leq n-1$, under the property of the uniform distribution

$$F_{\mathcal{U}}\left(\frac{|S_0|}{n-1}\right) = \frac{|S_0|}{n-1}$$

and the probability of the center switching is therefore

$$\mathbb{P}_i(G(n), S_0) = F_{\Theta_i}\left(\frac{|S_0|}{d_i(G(n))}\right) = F_{\Theta_i}\left(\frac{|S_0|}{n-1}\right) = F_{\mathcal{U}}\left(\frac{|S_0|}{n-1}\right) = \frac{|S_0|}{n-1}$$

For the "only if" part, fix some $|S_0| \in \mathbb{N}_+$ and $G(n)$, then as before

$$\mathbb{P}_i(G(n), S_0) = F_{\Theta_i}\left(\frac{|S_0|}{d_i(G(n))}\right) = F_{\Theta_i}\left(\frac{|S_0|}{n-1}\right)$$

Indeed, for each $1 \leq |S_0| \leq n-1$ on every $G(n)$, we require that

$$F_{\Theta_i}\left(\frac{|S_0|}{n-1}\right) = \frac{|S_0|}{n-1}$$

Hence, for each $1 \leq |S_0| \leq n-1$ on every $G(n)$, we require that:

$$F_{\Theta_i}\left(\frac{|S_0|}{n-1}\right) - F_{\Theta_i}\left(\frac{|S_0|-1}{n-1}\right) = \frac{1}{n-1}$$

Therefore we require that $\Pr(a \leq x \leq b) = b - a$ for any rational $a, b \in [0,1]$ s.t. $a < b$. Hence, we require that Θ_i is uniform over every open interval (a,b) for any rational $a, b \in [0,1]$ s.t. $a < b$ (there are countably infinitely many such intervals). Therefore, Θ_i is not uniform over a set of measure 0 so the distribution is uniform almost everywhere on $[0,1]$. □

It remains to show that in order to calculate the switching probability of i, we need to average the switching probability of $N_i(G)$ conditional on i not switching. We build on Claim 2.6 in Kempe et al. (2003). Without loss of generality, we can restrict ourselves to cases in which i in node that switches in the last round of the cascade process (what happens after i switches is irrelevant to i's probability of switching). Any live-edge path from the seeds that activates i must pass through at least one of the neighbors without passing through i first. Therefore, the probability of i switching must only depend on the probability of his neighbors switching, conditional on i not switching before. The probability that the live-edge path takes a step from a neigbor $j \in N_i(G)$ to i is $\frac{1}{d_i}$ irrespective of the path by from Lemma 1. The proposition follows immediately. □

Proof of Proposition 2. Again, we build on Kempe et al. (2003). By Claim 2.6, the distribution of switches in an LTM process is the same as the distribution of switches under the independent cascade process with live-edges. In an independent cascade process, by Claim 2.3, a node switches if and only if there is a live-edge path from some node in S_0. By construction of live-edge paths (i) this path avoids other nodes in S_0 (because nodes in S_0 are already activated) and (ii) the path is from exactly one node in S_0 (otherwise two path will activate one node). A node is activated by its neighbor along the live-edge path $P \in \mathcal{P}^*$ with probability $\frac{1}{d_i}$. Therefore, the probability that the nodes switches along this live-edge path is $\frac{1}{\chi_P}$. Hence, the probability that the node is activated by a path from a particular $j \in S_0$ is $\sum_{P \in \mathcal{P}} \frac{1}{\chi_P}$ and by Claim 2.3 the probability that the node is activated by any one $j \in S_0$ is $\sum_{j \in S_0} \sum_{P \in \mathcal{P}} \frac{1}{\chi_P}$.

□

Proof of Theorem 1. We first show that $\mathcal{C}_i(G) = d_i(G) + 1$ when G is a tree. To save on notation, we often refer to $\mathcal{C}_i(G)$ as $\mathcal{C}_i$.

We can show this by an induction on the number of nodes in the graph. For $n = 1$, $\mathcal{C}_i = 1$ since it is an isolated node and there is no path in the graph. Now, fix $N \geq 1$. For any $n \leq N$, we have $\mathcal{C}_i = 1 + d_i$. Now, consider $n = N + 1$. Fix a node $i \in V$. Then, we can decompose $\mathcal{C}_i$ as:

$$\mathcal{C}_i = 1 + \sum_{j \in N_i} \frac{1}{d_j} \cdot (\mathcal{C}_j(G) - 1).$$

However, $\mathcal{C}_j(G)$ can be re-expressed as $\mathcal{C}_j(G^{(j)})$ where $G^{(j)}$ is a branch of G starting from j. Now, each $G^{(j)}$ is a tree that has strictly smaller than n nodes. Hence, the induction hypothesis applies to each $G^{(j)}$, resulting in $\mathcal{C}_j(G) = \mathcal{C}_j(G^{(j)}) = d_j + 1$. Therefore, we conclude that:

$$\mathcal{C}_i = 1 + \sum_{j \in N_i} \frac{1}{d_j} \cdot (\mathcal{C}_j(G) - 1) = 1 + \sum_{j \in N_i} \frac{1}{d_j} \cdot ((d_j + 1) - 1) = 1 + \sum_{j \in N_i} 1 = d_i + 1.$$

Now that we have shown that when G is a tree $\mathcal{C}_i(G) = d_i(G) + 1$ holds, let us use it a base step for the next induction hypothesis. What we want to show is that the

contribution from loops and paths add up to $1 + d_i$:

$$\sum_{j \in V \setminus \{i\}} \sum_{P \in \mathcal{P}_{ij}} \frac{1}{\chi_P} + \sum_{j \in V} \sum_{L \in \mathcal{L}_{ij}} \frac{1}{\chi_L} = 1 + d_i$$

Suppose that Theorem 1 holds for a graph with $C > 0$ or fewer cycles. We want to show that Theorem 1 holds for a graph with $C + 1$ cycles.

The first two cases will allow us to generalize to deal with the most general Case (iii):

- Case (i): The seed is on a unique one cycle (i.e. a closed loop). Create a graph G' by removing all edges between seed i and $N_i(G)$ (as well as i) and attaching a leaf seed to every $N_i(G)$. The probability of switching for every node in G' that is also in G has not changed, but the size of the total contribution has increased by $d_i - 1$ because of the extra seeds. Consider G''_{d_i} are d_i copies of G' in which a unique seed of G' is a seed. The size of the contribution in G' (and G) is equal to the sum of the cascades in G''_{d_i}. By the induction hypothesis (since there are now fewer than C cycles), each G''_{d_i} has a contribution of 2 (since each seed has degree 1). Since there are d_i copies of it, the contribution of G is $2d_i - (d_i - 1) = d_i + 1$ as required. Note this argument also works even if there are cycles elsewhere in the graph (i.e. they do not include i).
- Case (ii): The seed is not on a cycle (but there are cycles elsewhere i.e. loops from i). Hence, the seed since on the intersection of one or more branches, each of which may lead to a cycle (if they do not, the branch is a tree and it's contribution is immediately 2). In this case note, that the contribution can be broken down into a the contribution of a unique path P (branch) to cycle and the contribution of a cycle. Subtracting the contribution of the path P, "move" the seed from i along the path to the cycle. The contribution of the cycle is 2 - the same as in case (i) - by the induction hypothesis (discounting by $\frac{1}{\chi_P}$).
- Case (iii): The seed is on multiple cycles and multiple branches. Apply case (i) to all cycles, creating $d_i - 1$ extra leaf seeds and case (ii) to all branches simultaneously. Using Case (i) or Case (ii), we have that the contribution of each d_i such G''_{d_i} with a single leaf seed is 2 by the induction hypothesis (as we have broken at least one cycle) because none of the probabilities of switching of any node have changed using our construction. Hence, once again we have $2d_i - (d_i - 1) = d_i + 1$ as required.

□

Proof of Proposition 3. The first statement is completely obvious and can be argued by contradiction.

That the maximum cascade is increasing up to $m = n - 1$ in m is also obvious (create the largest possible star at each stage).

To show the maximum cascade is decreasing from $m = n - 1$ in m, first note that in order to maximize the cascade with $m \geq n - 1$ edges it is necessary to have a node with $n - 1$ edges. If there is no such node, then it must be possible to rewire some

edges to this node. At each such rewiring, we will be increasing the cascade centrality of this node by Theorem 1 because its degree will be increasing and the contribution of loops will be falling. Therefore, for any $m \geq n-1$, the node that has the highest cascade centrality has degree $n-1$. Hence, any extra edge will necessarily create more loops (even allowing for rewiring) and by Theorem 1 the maximum cascade centrality will fall in m.

□

Proof of Theorem 2. Any realization of the deterministic cascade process $\kappa \in K$ on $G_{\boldsymbol{\theta}}$ is charactersized by a unique **sequence** of additional switches $\kappa(G_{\boldsymbol{\theta}}, S_0) = \{S_0, S_1, S_2, \ldots\}$. Any $\kappa(G_{\boldsymbol{\theta}}, S_0)$ can be represented by a directed tree $\tau(G, S_0) \in \mathcal{T}(G, S_0)$ rooted at s on a subgraph of $G_{\boldsymbol{\theta}}$, for which there is an edge from i to j if and only if $i \in S_t$ and $j \in S_{t+1}$. There are many trees that may represent the same sequence, but a given tree can represent at most one sequence.[12] V' switching can be represented by any such tree that includes V'.

Now let's consider the probability of this process when thresholds are uniform random.

Our aim is to show that the probability of precisely nodes V' switching can be decomposed into tree in which all incoming edges of i have weight $\frac{1}{d_i}$ for all i.

For a given κ, we say node is **exposed** in period t where $t = \min_{t \in 0,1,2,\ldots}\{t | S_t \in \kappa, S_t \cap N_i(G) \neq \emptyset\}$. Given a node is exposed in t, there are two cases:

1. $i \in S_{t+1}$: $S_t \cap N_i(G) = l$ $(l \geq 1)$. Using Assumption 1, probability that $i \in S_{t+1}$ is $\frac{l}{d_i}$. $\tau(G, S_0)$ representing κ can be extended with one of l possible edges. We attach a weight of $\frac{1}{d_i}$ any such extra edge.
2. $i \in S_{t+r}$: $S_t \cap N_i(G) = l$ $(l \geq 1)$ and $S_{t+r} \cap N_i(G) = l + a$. If a node switches when there are a **additional** neighbors who switch, then we know that most $\frac{l}{d_i} < \theta \leq \frac{l+a}{d_i}$. Using Assumption 1, the probability of the threshold being in that region is $\frac{a}{d_i}$. $\tau(G, j)$ representing κ in $t + r - 1$ can be extended with one (otherwise there is a cycle) of l possible edges. We attach a weight of $\frac{1}{d_i}$ any such extra edge.

Therefore each incoming edge into i in the tree has weight $\frac{1}{d_i}$ and the root has no incoming edges. Therefore, the weight of the tree is

$$\frac{1}{\chi_\tau}$$

Contagion is represented by a spanning tree rooted in i, the weight of which is $\frac{1}{\prod_{j \in V \setminus \{i\}} d_j}$. (The corresponding probability of the cascade process represented by all these trees is this probability times the number of trees that represent this cascade process). By Kirchoff's Theorem, the number of spanning trees of G (rooted in any node, since the

[12] Suppose not. A node switches if and only if its in the tree. A switched node can be member of only one tree τ by construction. Then suppose that a tree represents two cascade processes: in one agent i switches at time t and in the other i switches at time t'. But by construction the period in which i switches is equal to the length of the unique path in that tree from $s_\tau$to i. Therefore, $t = t'$, a contradiction.

number of spanning trees is same) is equal to the product of non-zero eigenvalues of the Laplacian over n. The result follows immediately.

□

Proofs of Propositions in Section 4

Proof of Corollary 1. **1.** See first part of proof of Proposition 2.

2. Fix a node i. Using Theorem 1, it suffices to subtract two loops from $1+2=3$, each contributing $1/2^{n-1}$. Therefore, $\mathcal{C}_i(G) = 3 - 2 \cdot 1/2^{n-1} = 3 - 1/2^{n-2}$. The limit result follows immediately.

3. Fix i and $j \neq i$. There are $\mathbf{P}(n-2, k-1)$ distinct paths of length k from i to j, each of which has a degree sequence product of $(n-1)^k$; the number of ways of permuting $k-1$ nodes upon selecting them out of $n-2$ possible candidates is precisely $\mathbf{P}(n-2, k-1)$.

We can therefore re-write the cascade centrality of any node in a complete graph $G(n)$ as:

$$\mathcal{C}_i(G(n)) = 1 + \sum_{i=0}^{n-1} \prod_{j=0}^{i} (1 - j/n - 1)$$

One can recognize this as the Ramanujan $Q(n)$-function. Following previous results, Flajolet et al. (1995) show that $Q(n)$ admits a full asymptotic expansion in descending powers of $\sqrt{n}$, so:

$$\mathcal{C}_i(G(n)) \sim 1 + \left[\sqrt{\frac{\pi n}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} - \frac{4}{135n} + \ldots \right]$$

This gives us the required result:

$$\lim_{n \to \infty} \frac{\mathcal{C}_i(G(n))}{\sqrt{n}} = \sqrt{\frac{\pi}{2}}.$$

Note also that we actually obtain an even tighter results: as $n \to \infty$, we have that $\mathcal{C}_i(G(n)) \to \sqrt{\frac{\pi n}{2}} + \frac{2}{3}$.

□

Proof of Proposition 4. We define a "regular" configuration model as follows:

Consider a degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers with an even sum and length n. $G_{\mathbf{d}}$ is a random (simple) graph with degree sequence $\mathbf{d}$ and its associated random configuration multigraph is $G^*_{\mathbf{d}}$ created by uniformly randomly matching on the degree half-edges (. It can be shown that the degree distribution in the configuration model converges in probability to the degree distribution of $G_{\mathbf{d}}$. Let $D = (r_0, r_1, \ldots) \in \mathcal{D}$ be a probability distribution on non-negative integers, such that $0 < \mathbb{E}(D) < \infty$. We say that the probability of realisation k denote random variable δ with probability distribution D is $r_k = \Pr(\delta = k)$. Finally, let $n_k(\mathbf{d}) = |\{j : d_j = k\}|$ denote the number of times a particular degree i occurs in $\mathbf{d}$ and $m(\mathbf{d}) = \frac{1}{2}\sum_{j=1}^{n} d_j$ as number of edges.

We impose two regularity conditions on D. We say that $\mathbf{d}_n \to D$ if

$$\lim_{n\to\infty} \frac{n_k(\mathbf{d}_n)}{n} = r_k$$

for each k and

$$\frac{m(\mathbf{d}_n)}{n} \to \frac{\mathbb{E}(D)}{2} = \frac{1}{2}\sum_{k=0}^{\infty} k r_k$$

as $n \to \infty$.

We say that a configuration model is "regular" if we let $D \in \mathcal{D}$ and assume that $\mathbf{d}_n \to D$.

This is enough to give us a general result for cascade centrality on random graphs.

Given a graph G, for $i \in V(G)$ and $\lambda \geq 0$, let $\Gamma_{\leq\lambda}(i) = \Gamma^G_{\leq\lambda}(i)$ denote the subgraph of G induced by the vertices within (graph) distance λ of i, regarded as a rooted graph with root i.

Lemma 2 (Corollary 5 in Bollobás and Riordan (2015))**.** *Suppose that $\mathbf{d}_n \to D$ and let $\lambda \geq 1$ be constant. Let i be a vertex of $G = G^*_{\mathbf{d}_n}$ chosen uniformly at random. Then with high probability*[13] *the neighborhood $\Gamma_{\leq t}(i)$ of i in G is a tree.*

The results follows immediately from Corollary 1 and from Lemma 2 by observing that Lemma 2 holds for any constant $\lambda \geq 1$.

□

Proof of Proposition 5. Loose bounds on the number of paths c_n (self-avoiding walks) of length n in an infinite $d - D$ hypercubic lattice, see equation 1.1.1 in Madras and Slade (1993):

$$d^n \leq c_n \leq 2d(2d-1)^{n-1}$$

The degree of each node in this lattice is $2d$ giving us for a generic node i:

$$\begin{aligned}
1 + \frac{r}{r-d} \leq \mathcal{C}_i &\leq \min\left\{1 + 2d, 1 + \frac{2d}{2d-1}\left(\frac{2d}{r-(2d-1)}\right)\right\} \\
1 + \frac{2d}{2d-d} \leq \mathcal{C}_i &\leq \min\left\{1 + 2d, 1 + \frac{4d^2}{2d-1}\right\} \\
3 \leq \mathcal{C}_i &\leq 1 + 2d
\end{aligned}$$

since $d > 1$.

□

Proof of Corollary 2. **1.** Tree: $\mathcal{C}_i(G)$ follows immediately from Proposition 2.

$$\begin{aligned}
\mathcal{K}(G) &= \frac{\left[\frac{1}{\Pi_{i\in V\setminus s_1} d_i} + \ldots + \frac{1}{\Pi_{i\in V\setminus s_n} d_i}\right]}{n} \\
&= \frac{d_1 + \ldots + d_N}{n \times \Pi_{i\in V} d_i} = \frac{\sum_{i\in V} d_i}{n \times \Pi_{i\in V} d_i} = \frac{2(n-1)}{n \times \Pi_{i\in V} d_i}
\end{aligned}$$

[13] Given a sequence E_n of events, E_n holds with high probability if $\Pr(E_n) \to 1$ as $n \to \infty$.

2. For any cycle, there are n spanning trees rooted in any node. Since $d_i = 2$ for all $i \in V$

$$\mathcal{K}_i(G) = \mathcal{K}(G) = \frac{n \times 2n}{n \times 2^n} = \frac{n}{2^{n-2}}$$

3. Using the Cayley formula:

$$\mathcal{K}_i(G) = \mathcal{K}(G) = \frac{n^{n-2}}{(n-1)^{n-1}} =$$
$$= \frac{n^{n-1}}{n \times (n-1)^{n-1}} = \frac{1}{n}\left(\frac{n-1+1}{n-1}\right)^{n-1} = \frac{1}{n}\left(1 + \frac{1}{n-1}\right)^{n-1}$$

and the limit as $n \to \infty$ of the term in brackets is e.

□

Proofs of Propositions in Section 5

Proof of Proposition 7. Note that $\rho(1-\rho)$ is strictly decreasing function for $\rho \in [1/2, 1]$. Also, $a_1 + a_2(1-\rho) + \cdots + a_n(1-\rho)^n$ is strictly decreasing function for $\rho \in [1/2, 1]$. □

Proof of Proposition 8. Fix a seed node i. Then, a profit can be written as $\pi(i, \rho) = \rho(a_1(1-\rho) + \cdots + a_D(1-\rho)^D) \leq \rho(1-\rho)\sum_{i=1}^{D} a_i = \rho(1-\rho)(\mathcal{C}_i - 1)$.

Now, we already know that $\mathcal{C}_i(G) \leq n$, with equality if and only if G is a star.

Hence, center i^* gives the highest profit. So,

$$\begin{aligned} \pi(i^*, \rho, n) &= (1-\rho)(n-1)\rho \\ \frac{\partial \pi}{\partial \rho} &= (n-1)(1-\rho-\rho) = 0 \\ \rho^* &= \frac{1}{2} \end{aligned}$$

Hence

$$\pi(i^*, \rho^*, n) = \frac{1}{4}(n-1)$$

□

Proof of Proposition 9. We prove each item separately.

1. Profit ranking of nodes and optimal seed $i \in \{1, 2, 3, \ldots\}$

Note that as $n \to \infty$

$$\pi(2, \rho) = \rho\left(\underbrace{(1-p)}_{\text{node 1}} + \underbrace{\frac{(1-\rho)}{2}}_{\text{node 3}} + \underbrace{\left(\frac{1-\rho}{2}\right)^2}_{\text{node 4}} + \ldots\right)$$

and

$$\pi(3,\rho) = \rho\left(\underbrace{\frac{(1-\rho)^2}{2}}_{\text{node 1}} + \underbrace{\frac{(1-\rho)}{2}}_{\text{node 2}} + \underbrace{\frac{(1-\rho)}{2}}_{\text{node 4}} + \underbrace{\left(\frac{1-\rho}{2}\right)^2}_{\text{node 5}} + \dots\right)$$

Note that $\pi(2,\rho) > \pi(3,\rho)$ for $\rho \in (0,1)$. So, $\pi(3,\rho_3^*) < \pi(2,\rho_3^*) \leq \pi(2,\rho_2^*)$ where ρ_i^* is the optimal price when ith node is a seed. So, $\pi(2,\rho_2^*) > \pi(3,\rho_3^*)$.

Similarly,

$$\pi(4,\rho) = \rho\left(\underbrace{\frac{(1-\rho)^3}{4}}_{\text{node 1}} + \underbrace{\frac{(1-\rho)^2}{4}}_{\text{node 2}} + \underbrace{\frac{(1-\rho)}{2}}_{\text{node 3}} + \underbrace{\frac{(1-\rho)}{2}}_{\text{node 5}} + \underbrace{\left(\frac{1-\rho}{2}\right)^2}_{\text{node 6}} + \dots\right)$$

Hence, $\pi(4,\rho) < \pi(3,\rho)$. So, similarly proceeding, we find that $\pi(4,\rho_4^*) < \pi(3,\rho_3^*)$. Therefore, $\pi(i,\rho_i^*) > \pi(i+1,\rho_{i+1}^*)$ for each $i < n/2$.

Finally,

$$\pi(1,\rho) = \rho\left(\underbrace{\frac{(1-\rho)}{2}}_{\text{node 2}} + \underbrace{\left(\frac{1-\rho}{2}\right)^2 + \dots}_{\text{node 3}}\right)$$

Hence, $\pi(1,\rho) < \pi(i,\rho)$ for each $i = 2, \cdots, \lfloor n/2 \rfloor$. Thus, $\pi(1,\rho_1^*)$ is the smallest possible profit.

2. Optimal price at the optimal seed

Seeding $i^* \in \{2, n-1\}$ and as $n \to \infty$

$$\pi(i^*,\rho) = \rho \cdot ((1-\rho) + \frac{(1-\rho)}{2} + \left(\frac{1-\rho}{2}\right)^2 + \cdots) = \frac{2\rho}{1+\rho} - \rho^2.$$

Its derivative is:

$$-\frac{2(\rho^3 + 2\rho^2 + \rho - 1)}{(1+\rho)^2}.$$

Its real root is:

$$\rho^* = \frac{1}{6}(-4 + \sqrt[3]{4} \cdot \sqrt[3]{29 - 3\sqrt{93}} + \sqrt[3]{4} \cdot \sqrt[3]{29 + 3\sqrt{93}} \approx .465571$$

Hence, the optimal profit is $\approx .418588$.

3. Optimal price at the leaf seed

For a leaf seed i and a line of length n

$$\pi(i,\rho,n) = \rho \times \left[\frac{2 - 2\rho\left(\frac{1-\rho}{2}\right)^{n+1}}{1+\rho} - 1\right]$$

As $n \to \infty$

$$\begin{aligned}
\pi(i,\rho) &= \rho \times \left[\frac{2}{1+\rho} - 1\right] \\
\frac{\partial \pi}{\partial \rho} &= -\frac{\rho^2 + 2\rho - 1}{(\rho+1)^2} = 0 \\
\rho^* &= \begin{cases} \sqrt{2} - 1 & > 0 \\ -1 - \sqrt{2} & < 0 \end{cases}
\end{aligned}$$

SOC is < 0.

4. Immediate from the expression of $\pi(i,\rho)$ that is increasing in n.

□

Proof of Proposition 10. For a generic seed i and a cycle of length n

$$\pi(i,\rho,n) = \rho \times \left[2 \times \frac{1 - \left(\frac{1-\rho}{2}\right)^n}{1 - \frac{1-\rho}{2}} - 1\right]$$

As $n \to \infty$

$$\begin{aligned}
\pi(i,\rho) &= \rho \times \left[\frac{2}{1 - \frac{1-\rho}{2}} - 1\right] \\
\frac{\partial \pi}{\partial \rho} &= -\frac{2(\rho^2 + 2\rho - 1)}{(\rho+1)^2} = 0 \\
\rho^* &= \begin{cases} \sqrt{2} - 1 & > 0 \\ -1 - \sqrt{2} & < 0 \end{cases}
\end{aligned}$$

SOC is < 0. Note that that asymtotically this it the same price at the optimal price for a leaf seed on an infinite line.

□

Proof of Proposition 11.

Lemma 3. *Fix* $0 < \rho < 1$ *and let*

$$Q(n,\rho) = \sum_{k=0}^{\infty} a_k$$

where

$a_0 = 1$, $a_1 = \left(1 - \frac{1}{n}\right)(1-\rho)$, $a_2 = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)(1-\rho)^2$, $a_3 = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)(1-$

etc. Then, $Q(n,p) \to \frac{1}{\rho}$ *as* $n \to \infty$.

Proof of Lemma 3. The kth term in the sum is:

$$a_k = \left(1 - \frac{1}{n}\right) \cdot \cdots \cdot \left(1 - \frac{k}{n}\right) (1-\rho)^k.$$

First of all,

$$a_k \leq 1 \cdot \cdots \cdot 1 \cdot (1-\rho)^k = (1-\rho)^k$$

.

So,

$$Q(n,\rho) \leq \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} p^k = \frac{1}{\rho}$$

.

Also,

$$a_k \geq \left(1 - \frac{k}{n}\right) (1-\rho)^k$$

Hence,

$$Q(n,\rho) \geq \sum_{i=0}^{k} \left(1 - \frac{k}{n}\right)^i (1-\rho)^i.$$

Now, as $n \to \infty$, we get $\sum_{i=0}^{k} (1-\rho)^i$ for a fixed constant k. Now, $Q(n,\rho) \geq \sum_{i=0}^{k} (1-\rho)^i$ for any k. Hence, because $Q(n,p) \leq \frac{1}{\rho}$, it follows that $Q(n,\rho) \to \frac{1}{\rho}$ by the sandwich theorem as $n \to \infty$. □

By stochastic cascade centrality tends to set $\frac{1}{\rho}$ so there are

$$\frac{1}{\rho} - 1$$

extra adopters i.e. the size of the cascade no longer depends on n, but only on the price. Hence, as $n \to \infty$ the firm's profit tends to

$$\rho(\frac{1}{\rho} - 1) = 1 - \rho$$

and, since by Lemma 3 ρ is bounded away from 0, profit is maximized when ρ is arbitarily close to 0. □

Proof of Proposition 12. For a generic seed i,

$$\begin{aligned}
\pi(i,\rho) &= \rho \cdot \left(\frac{1-\rho}{r} \cdot r + \frac{(1-\rho)^2}{r^2} \cdot r(r-1) + \frac{(1-\rho)^3}{r^3} \cdot r(r-1)^2 \cdots\right) \\
&= \rho \cdot \frac{r}{r-1} \cdot \left((1-\rho) \cdot \frac{r-1}{r} + (1-\rho)^2 \cdot \frac{(r-1)^2}{r^2} + \cdots\right) \\
&= \rho \cdot \frac{r}{r-1} \cdot \left(\frac{1}{1-(1-\rho)(1-1/r)} - 1\right)
\end{aligned}$$

Hence, $\frac{\partial \pi}{\partial p} = \frac{r((r-1)\rho^2 + 2\rho - 1)}{((r-1)\rho + 1)^2}$, which which means that $\rho^* = \frac{1}{1 \pm \sqrt{r}}$. Since $\rho^* \in [0, 1]$, we have that

$$\rho^* = \frac{1}{1 + \sqrt{r}}.$$

And,

$$\pi(i, \rho^*) = \frac{r}{(\sqrt{r} + 1)^2}.$$

□

Proofs of Propositions in Section 6

Proof of Proposition 13 . Define $\frac{1}{\chi^{P_{ii}}} = 1$.

$$\begin{aligned}
\phi_i(G) &= \frac{1}{n}\left[\sum_{j \in N}\sum_{P_{ji} \in \mathcal{P}_{ji}} \frac{1}{\chi^{P_{ji}}}\right] \\
&= \frac{1}{n}\left[\sum_{j \in N}\sum_{P_{ji} \in \mathcal{P}_{ji}} \frac{1}{d_k \times \ldots \times d_i}\right] \\
&= \frac{1}{n}\left[\sum_{j \in N}\sum_{P_{ji} \in \mathcal{P}_{ji}} \frac{d_j}{d_j \times \ldots \times d_k \times \ldots \times d_i}\right] \\
&= \frac{1}{n}\left[\sum_{j \in N}\sum_{P_{ji} \in \mathcal{P}_{ji}} \frac{1}{d_i} \times \frac{d_j}{d_k \times \ldots \times d_j}\right] \\
&= \frac{1}{d_i n}\left[\sum_{j \in N}\sum_{P_{ij} \in \mathcal{P}_{ij}} \frac{d_j}{\chi^{P_{ij}}}\right]
\end{aligned}$$

If G is regular then

$$d_i = d_j = d$$

hence

$$\begin{aligned}
\phi_i(G) &= \frac{1}{dn}\left[\sum_{j \in N}\sum_{P_{ij} \in \mathcal{P}_{ij}} \frac{d}{\chi^{P_{ij}}}\right] \\
&= \frac{d}{dn}\left[\underbrace{\sum_{j \in N}\sum_{P_{ij} \in \mathcal{P}_{ij}} \frac{1}{\chi^{P_{ij}}}}_{\mathcal{C}_i(G)}\right]
\end{aligned}$$

□

Proof of Proposition 14 . Probability of failure for a center link where there are n nodes and m edges:

$$\begin{aligned}\phi_i(G) &= \frac{1}{n} + \frac{1}{n}\left[\frac{m}{m}\right] \\ &= \frac{2}{n}\end{aligned}$$

which is independent of m. Hence, the center will want to keep all his edges if $\delta \geq c$ (otherwise none).

Probability of faiure of a leaf in a star with $n-1$ edges

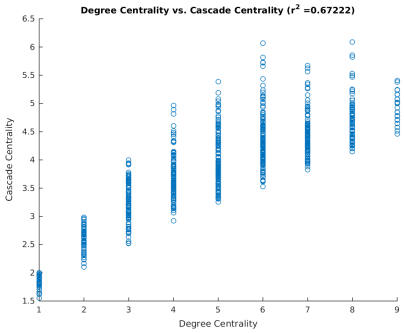
$$\begin{aligned}\phi_i(G) &= \frac{1}{n} + \frac{1}{n} + \frac{1}{n}\left[\frac{n-2}{n-1}\right] \\ &= \frac{3n-4}{n(n-1)}\end{aligned}$$

Probability of failure of leaves in a star that have just added an edge between them:

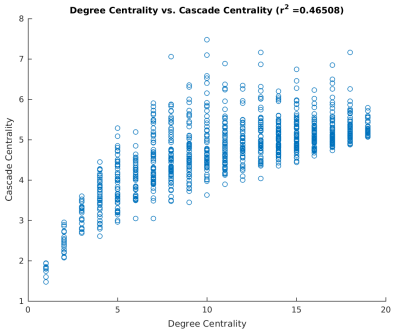
$$\begin{aligned}\phi_i(G) &= \frac{1}{n} + \frac{1}{n}\left[\frac{1}{2} + \frac{1}{4}\right] + \frac{n-3}{n}\left[\frac{1}{2(n-1)} + \frac{1}{4(n-1)}\right] + \frac{1}{n}\left[\frac{1}{2(n-1)} + \frac{1}{2}\right] \\ &= \frac{3n-4}{n(n-1)}\end{aligned}$$

Hence, the leaves won't form an edge only if $c < \delta$. A contradiction.

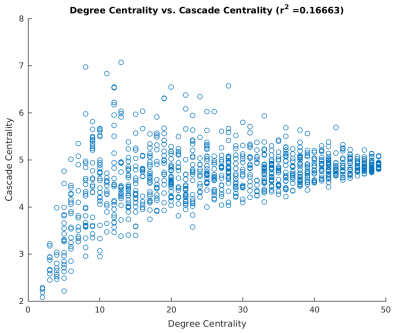
□



(a) $n = 10$

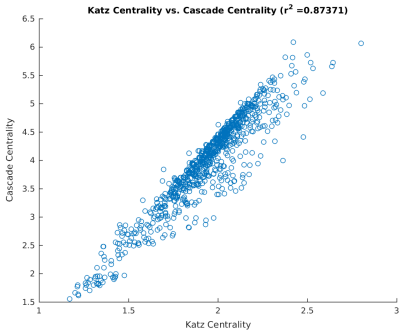


(b) $n = 20$

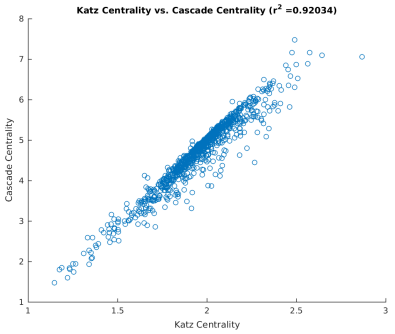


(c) $n = 50$

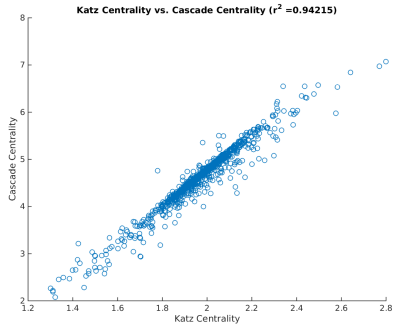
Figure 8: Cascade centrality vs. degree centrality



(a) $n = 10$



(b) $n = 20$

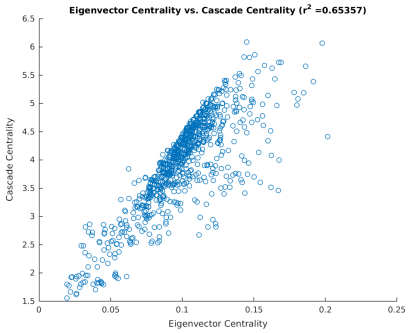


(c) $n = 50$

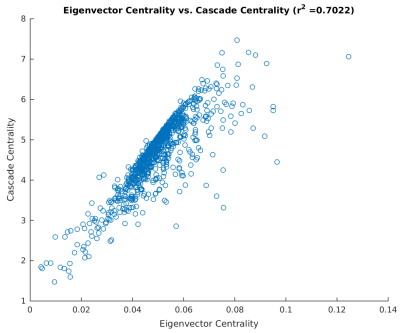
Figure 9: Cascade centrality vs. Katz centrality

Appendix B: Simulations

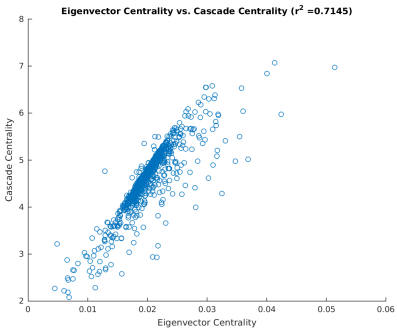
We considered connected networks of size $n = 10, 20, 50$. We uniformly randomly selected a number of edges $n - 1 \leq m \leq \binom{n}{2}$ for each graph size and then randomly selected a network with m edges with a nonsingular adjacency matrix. We then randomly selected one node from this network. We repeated this 1000 times. For Katz centrality, we fixed α to be half of the largest eigenvalue of the adjacency matrix. k-order approximation means we apply Theorem 1 only to paths of length k. In networks of size 10, cascade centrality is computed exactly; in networks of size 20, it is computed up to paths of length 5; in networks of size 50, it is computed up to paths of length 4.



(a) $n = 10$

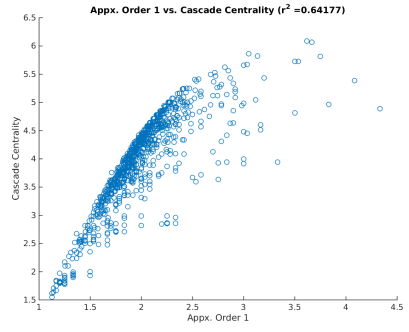


(b) $n = 20$

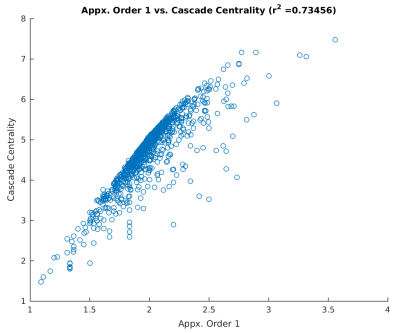


(c) $n = 50$

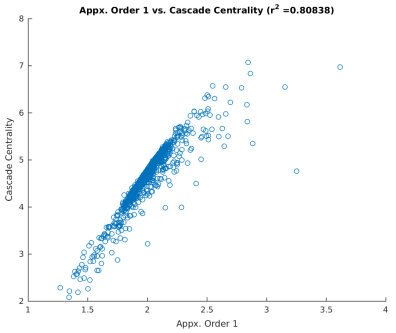
Figure 10: Cascade centrality vs. eigenvector centrality



(a) $n = 10$

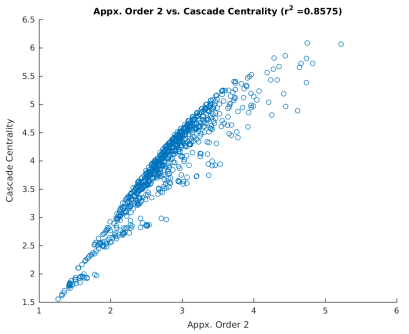


(b) $n = 20$

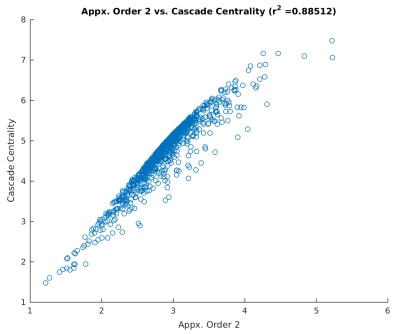


(c) $n = 50$

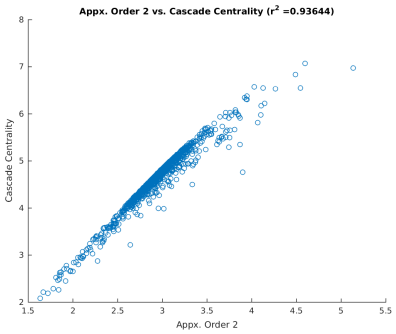
Figure 11: Cascade centrality vs. 1st order approximation



(a) $n = 10$

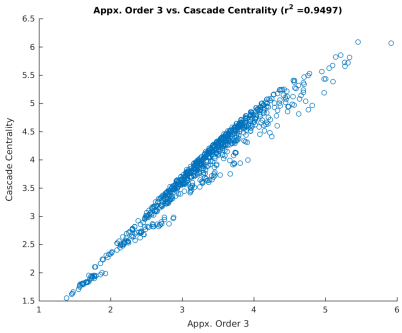


(b) $n = 20$

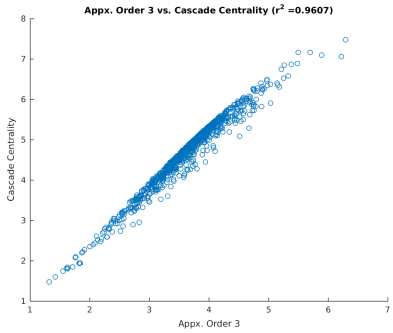


(c) $n = 50$

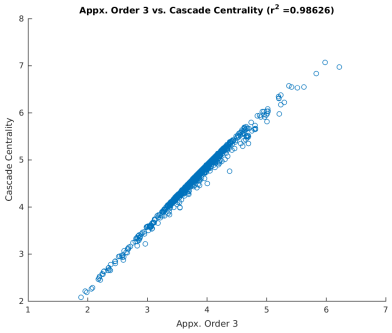
Figure 12: Cascade centrality vs. 2nd order approximation



(a) $n = 10$



(b) $n = 20$



(c) $n = 50$

Figure 13: Cascade centrality vs. 3rd order approximation

Appendix C: Macroscopic comparative statics

Figure 14 illustrates that increased clustering that either increase or decrease average cascade centrality across all nodes (and therefore the cascade centrality of at least one node).

Higher clustering $\Rightarrow$ lower cascade centrality. Graphs A and B have 4 nodes and 4 edges. The global clustering coefficient in A is 0 and in B is 0.6. Average cascade centrality across all nodes is 2.75 in A and $2\frac{2}{3}$ in B.

Higher clustering $\Rightarrow$ higher cascade centrality. Graphs C and D have 5 nodes and 5 edges. The global clustering coefficient in C is $\frac{3}{8}$ and in D is $\frac{3}{7}$. Average cascade centrality across all nodes is 2.75 in C and $2\frac{7}{9}$ in D.

Higher clustering $\Rightarrow$ higher cascade centrality for a node of a fixed degree. Graphs E and F have 5 nodes and 5 edges. The global clustering coefficient in E is $\frac{3}{7}$ and in F is $\frac{1}{2}$. The cascade centrality of node i is $1\frac{8}{9}$ in E and $1\frac{11}{12}$ in F.

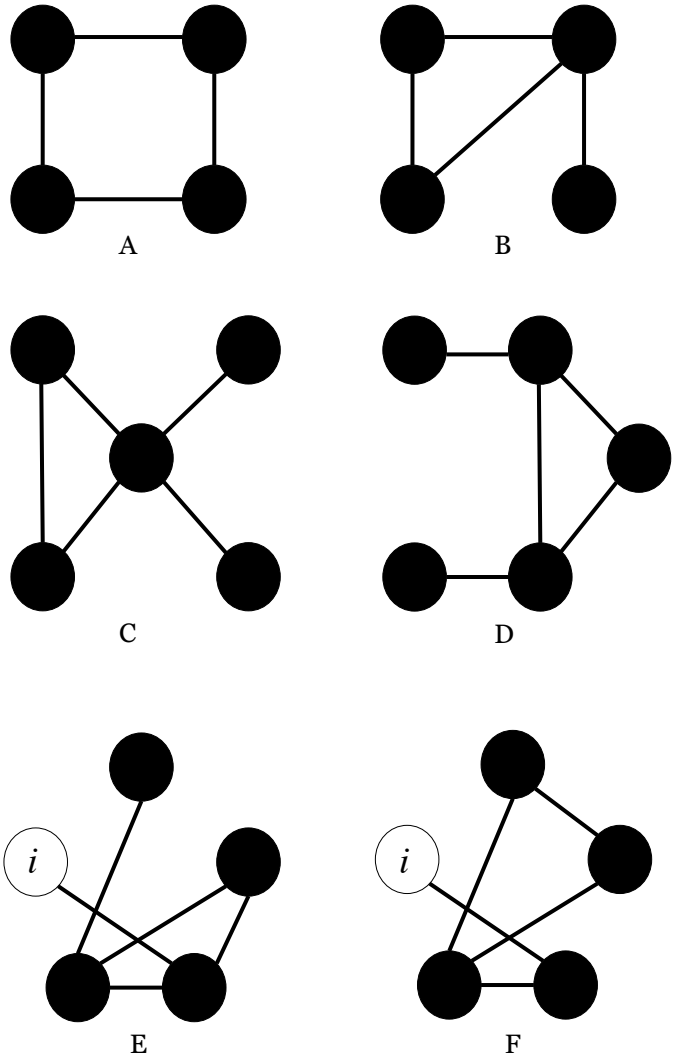


Figure 14: Clustering and cascade centrality

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